

Lie algebras associated with one-dimensional aperiodic point sets

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Abstract

The set of points of a one-dimensional cut-and-project quasicrystal or model set, while not additive, is shown to be multiplicative for appropriate choices of acceptance windows. This leads to the definition of an associative additive graded composition law and permits the introduction of Lie algebras over such aperiodic point sets. These infinite dimensional Lie algebras are shown to be representatives of a new type of semi-direct product induced Lie algebras.

1 Introduction

In the study of aperiodic structures, the *Fibonacci chain* \mathcal{F} , a one-dimensional aperiodic sequence, plays an important role. It may be co-ordinatized by the sequence $\{\mathcal{F} : \dots, -4\tau - 2, -3\tau - 1, -2\tau - 1, -\tau, 1, \tau + 1, 2\tau + 2, 3\tau + 2, 4\tau + 3, 5\tau + 4, 6\tau + 4, 7\tau + 5, 8\tau + 5, 9\tau + 6, 10\tau + 7, \dots\}$, which is given for all positive and negative integers $n_2 \in \mathbb{Z}$ by

$$\mathcal{F}(n_2) = n_2\tau + \left\lfloor \frac{n_2}{\tau} + 1 \right\rfloor. \quad (1)$$

The floor function, denoted by $\lfloor x \rfloor$, is the greatest integer less than, or equal to x ; whilst $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ denotes the *golden mean*, which satisfies the equation

$$x^2 = x + 1. \quad (2)$$

It corresponds to the sequence featured in, e.g., [1, 2, 3].

The Fibonacci chain \mathcal{F} in (1) is a subset of the ring $\mathbb{Z}[\tau] := \mathbb{Z} + \mathbb{Z}\tau$, equipped with the Galois automorphism that maps $n = n_1 + n_2\tau \in \mathbb{Z}[\tau]$ to $n^* = n_1 + n_2\tau'$, where $\tau' \equiv -\tau^{-1}$ is the algebraic conjugate¹ of τ . \mathcal{F} is an example of a *cut-and-project quasicrystal* or *model set*, i.e., a set of the form

$$\Sigma(\Omega) = \{n \in \mathbb{Z}[\tau] \mid n^* \in \Omega\}, \quad (3)$$

and $\Omega \subset \mathbb{R}$ is a connected interval, called an *acceptance window*. In particular, the Fibonacci chain \mathcal{F} amounts to $\Sigma((0, 1])$. While (1) is useful for explicit evaluation of the points in \mathcal{F} , set theoretic properties of the chain can be discussed more conveniently in terms of (3).

Recently, a new class of infinite dimensional Lie algebras was introduced [5], which possess indices valued in cyclotomic fields $\mathbb{Z}(\omega)$ with $\omega^N = 1$ an N -th root of unity. For $n, m \in \mathbb{Z}(\omega)$ and $a, b \in \{0, \dots, N-1\}$, these algebras are defined by the relation

$$[J_m^a, J_n^b] = J_{m+w^a n}^{a+b} - J_{n+w^b m}^{a+b}, \quad (4)$$

¹It is the other root of the defining quadratic equation (2), and so the multiplication table of functions of τ' is identical to that of functions of τ .

and may be constructed from the more succinct associative product law

$$J_m^a J_n^b = J_{m+wa_n}^{a+b}. \quad (5)$$

This is an algebra in which one of the indices adds with a phase factor, unlike the conventional indices of Kac-Moody algebras.

For $N \geq 5$ the set $\mathbb{Z}(\omega)$ is dense in \mathbb{R}^2 , and so is any one-dimensional subset. We consider here such one-dimensional subsets, and introduce a condition to select appropriate uniformly discrete subsets of these sets. In particular, since $\tau = -(\omega^2 + \omega^3)$ for $\omega^5 = 1$, the sets $\Sigma(\Omega)$ in (3) are, in fact, uniformly discrete one-dimensional subsets of $\mathbb{Z}(\omega)$ that are located on the real line in the complex plane.

A crucial feature of the sets $\Sigma(\Omega)$ is their lack of additivity, which does not encourage trivial applications. However, it is shown here that, for appropriate choices of Ω , the set is *multiplicative*. Moreover, a further, associative, graded additive composition closure holds: For all $m, n \in \Sigma(\Omega)$ there exists $a \in \mathbb{N}$ such that $m + \tau^a n \in \Sigma(\Omega)$. In particular, multiplicativity allows one to restrict (4) to a subalgebra (Lie algebra) (5), with generators in a one-to-one correspondence with points in a one-dimensional aperiodic point set.

Note that these Lie algebras are different from the known family of aperiodic Virasoro algebras constructed over $\Sigma(\Omega)$, proposed in [2, 3]. Specifically, in these references, an aperiodic analog to the Virasoro algebra has been defined via a replacement of the usual index set \mathbb{Z} by a cut-and-project quasicrystal $\Sigma([a, b])$ with $a \cdot b \geq 0$. The corresponding Lie algebras are given by the relation

$$[L_n, L_m] = (m - n)\chi_{[a,b]}(n' + m')L_{n+m}, \quad (6)$$

where $\chi_{[a,b]}$ denotes the characteristic function of the set $[a, b]$ [2]. It has been shown that a nontrivial central extension exists for the case of $[a, b] = [0, 1]$, if the structure constants are restricted to the τ -component of the points $(n_1 + \tau n_2) \in \Sigma([0, 1])$, i.e. to n_2 [3]. Note that these algebras are by construction nilpotent, due to the factor $\chi_{[a,b]}$. By contrast, the algebras defined in this paper are not nilpotent.

2 Algebras reliant on multiplicative closure of $\Sigma(\Omega)$

This section relies on the following theorem:

Theorem II.1 $\Sigma(\Omega)$ in (3) is closed under multiplication if and only if Ω is a connected subset of $[-1, 1]$ of the form $[-\alpha, \beta]$, $0 \leq \alpha \leq 1$ and $\beta \geq \alpha^2$, open boundaries being also permissible.

Proof. Let $(n_1 + n_2\tau), (m_1 + m_2\tau) \in \Sigma(\Omega)$, with Ω as stated in the theorem. By definition, $(n_1 + n_2\tau'), (m_1 + m_2\tau') \in \Omega$. Since the multiplication properties of τ and τ' are identical, the Galois conjugate of this product is the product of the conjugates of the factors,

$$(n_1 + n_2\tau')(m_1 + m_2\tau') = ((n_1 + n_2\tau)(m_1 + m_2\tau))' \in \Omega. \quad (7)$$

Thus, $(n_1 + n_2\tau)(m_1 + m_2\tau) \in \Sigma(\Omega)$. This proves that the set is closed under multiplication. ■

In particular, it follows that the Fibonacci chain \mathcal{F} is multiplicative². Since $(\tau')^{2m} \in (0, 1]$, $(\tau')^{2m-1} \in [-1, 0)$, and $(\tau')^m \in [-1, 1]$ for $m \in \mathbb{N}$, the point sets $P_1 := \{\tau^{2m} \mid m \in \mathbb{N}\}$, $P_2 := \{\tau^{2m-1} \mid m \in \mathbb{N}\}$ and $P_1 \cup P_2$ are examples of multiplicative subsets of \mathcal{F} , $-\mathcal{F}$ (i.e. the sequence obtained after reflection at the origin) and $\Sigma([-1, 1])$, respectively.

The largest set Ω for which $\Sigma(\Omega)$ is closed under multiplication is $\Omega = [-1, 1]$, and we will therefore construct algebras over elements in this set. Algebras related to subsets of $\Sigma(\Omega)$ that are also closed under multiplication correspond to subalgebras of this algebra.

²Specifically, $\forall m_2, n_2 \in \mathbb{Z}$, and for $p_2 \equiv m_2 n_2 + n_2 \lfloor \frac{m_2}{\tau} + 1 \rfloor + m_2 \lfloor \frac{n_2}{\tau} + 1 \rfloor$, it follows that $mn = p \in \mathcal{F}$. The Fibonacci chain \mathcal{F} is an Abelian monoid. Another, technical, way to cast this result of the theorem is as an identity for the floor function,

$$\left\lfloor \frac{1}{\tau} \left(m_2 n_2 + m_2 \left\lfloor \frac{n_2}{\tau} + 1 \right\rfloor + n_2 \left\lfloor \frac{m_2}{\tau} + 1 \right\rfloor \right) + 1 \right\rfloor = m_2 n_2 + \left\lfloor \frac{m_2}{\tau} + 1 \right\rfloor \left\lfloor \frac{n_2}{\tau} + 1 \right\rfloor.$$

Due to the property

$$\Sigma(\Omega) = -\Sigma(-\Omega) , \quad (8)$$

the set $\Sigma([-1, 1])$ decomposes as

$$\Sigma([-1, 1]) = -\Sigma((0, 1]) \oplus \{0\} \oplus \Sigma((0, 1]), \quad (9)$$

where $\Sigma((0, 1])$ again corresponds to the Fibonacci chain \mathcal{F} .

Let $m, n \in \Sigma([-1, 1])$. Then $mn \in \Sigma([-1, 1])$, and hence the *logarithmic set over* $\Sigma([-1, 1])$, defined as

$$\mathcal{L} \equiv \{\log(|m|) \mid m \in \Sigma([-1, 1])\} \quad (10)$$

is additive, with $\log(|m|) + \log(|n|) = \log(|mn|)$. Thus the logarithms of the set behave additively, and can then index an alternative centerless Virasoro algebra on the quasicrystal to that proposed in [2, 3]:

$$[L_{\log(|m|)}, L_{\log(|n|)}] = \log\left(\frac{|m|}{|n|}\right) L_{\log(|mn|)} . \quad (11)$$

(One may visualize this through a conventional Witt algebra realization $L_{\log(|m|)} = |m|^{-x} \partial_x$.)

Given the existence of primes on this set, e.g., $5\tau + 4$, the multiplicative composition of indices presents a peculiarity, illustrated by the analogous multiplicative centerless Virasoro algebra over the integers. Writing the n th element as l_n (substituting $m \in \mathbb{Z}$ for $\log(|m|)$ etc.),

$$[l_m, l_n] = (m - n) l_{mn} . \quad (12)$$

The unusual feature is that if p is prime, then l_p cannot be constructed as the commutator of two other elements in the algebra, and neither can be l_p^2 .

Remark II.2 *It is straightforward to extend the definition of cut-and-project quasicrystals or model sets to irrational numbers defined as solutions α of the quadratic equation $x^2 = mx + 1$, $m \in \mathbb{N}$, and $x^2 = mx - 1$, $m \in \mathbb{N}$, $m \geq 3$, (with Galois conjugate α'), i.e., to point sets*

$$\Sigma_\alpha(\Omega) := \{n \in \mathbb{Z}[\alpha] \mid n^* = n_1 + \alpha' n_2 \in \Omega\} , \quad (13)$$

where $\mathbb{Z}[\alpha] = \mathbb{Z} + \mathbb{Z}\alpha$. Theorem II.1 holds *mutatis mutandis*, for Ω appropriate subsets of $[-1, 1]$. The sets $\Sigma_\alpha([-1, 1])$ are also closed under multiplication and define algebras along the lines given above.

Remark II.3 *The proof above relies on the set-theoretic properties of $\Sigma(\Omega)$. Alternatively, the claim may be proven for the Fibonacci chain \mathcal{F} directly starting from the explicit floor function definition and its constrained minimization of quadratic integer polynomials.*

Remark II.4 *In an interesting paper Berman and Moody [4] introduce a completely different (non-associative) addition law; $m \vdash n = \tau^2 m - \tau n$ which has the property that if $m, n \in \Sigma$ so is $\tau^2 m - \tau n$. Thus this property together with the multiplicative law makes Σ into a non-associative ring.*

3 Additive Grading in $\Sigma([-1, 1])$

As shown in Theorem II.1, the set $\Sigma([-1, 1])$ is maximal with respect to multiplicative closure, and thus plays a distinguished role. Moreover, by virtue of (9), its properties follow from the properties of the Fibonacci chain, i.e., from $\Sigma((0, 1])$. We now further consider graded addition closure of this set.

Lemma III.1 *Let $n \in \Sigma((0, 1])$. If $n \in \Sigma((0, \frac{1}{\tau}])$, then $n + \tau^2 m \in \Sigma((0, 1])$, while if $n \in \Sigma((\frac{1}{\tau}, 1])$, then $n + \tau m \in \Sigma((0, 1])$.*

Proof. Let $n \in \Sigma((0, 1])$. Then, either $n \in \Sigma((\frac{1}{\tau}, 1])$, or $n \in \Sigma((0, \frac{1}{\tau}])$. Consider first $n \in \Sigma((\frac{1}{\tau}, 1])$; by definition, $n^* \in (\frac{1}{\tau}, 1]$. Hence, $(n + \tau m)^* = n^* + \tau' m^* \in (0, 1]$, $\forall m \in \Sigma((0, 1])$, and thus $n + \tau m \in \Sigma((0, 1])$, $\forall m \in \Sigma((0, 1])$. The claim for $n \in \Sigma((0, \frac{1}{\tau}])$ follows analogously. ■

Lemma III.2 *The partition of $\Sigma((0, 1])$ in Lemma III.1 is exclusive, i.e., for every $n \in \Sigma((0, 1])$, one has either $n + \tau m \in \Sigma((0, 1])$ or $n + \tau^2 m \in \Sigma((0, 1])$, $\forall m \in \Sigma((0, 1])$.*

Proof. Let $n, m \in \Sigma((0, 1])$. If $n^* + \tau' m^* \notin (0, 1]$, then n^* must lie in $(0, \frac{1}{\tau}]$, as minimum $n^* + \tau' m^* < 0$. But then the above argument shows

that $n^* + \tau'^2 m^* \in (0, 1]$. Likewise, if $n^* + \tau' m^* \notin (0, 1]$, then $n^* \in (\frac{1}{\tau}, 1]$, as maximum $n^* + \tau' m^* > 1$. ■

Due to Lemma III.2 *all* elements n of $\Sigma((0, 1])$ may be assigned either an even or an odd “parity”, depending on inclusion of their conjugates n^* in the respective intervals $(0, 1/\tau]$ or $(1/\tau, 1]$. Lemma III.1 implies that addition of an odd (even) element with τ (τ^2) times any element of $\Sigma((0, 1])$ produces elements of $\Sigma((0, 1])$ ³.

Remark III.3 *Depending on the choice of the boundaries of the acceptance window Ω in (3), the occurrence of an exceptional tile is possible, i.e. a tile whose nearest neighbour distance (tile) in the chain occurs precisely once. For example, in the case of the Fibonacci chain \mathcal{F} , i.e. for $\Sigma((0, 1])$, all tiles are of the size τ or τ^2 . However, in $\Sigma([0, 1])$, there occurs, in addition, an exceptional tile of size 1, located between the points 0 and 1.*

The occurrence of exceptional tiles in a point set $\Sigma(\Omega)$ has implications on the assignment of parity to the points. In particular, in the presence of an exceptional tile there occurs precisely one point to which parity cannot be assigned in a unique way: it is that point, which, after removal of one of the boundary points of Ω , lies on the boundaries (open and closed respectively) of the intervals that partition Ω and determine the parity. For example, for $\Sigma([0, 1])$, it is the point $n = -\tau$ with $n^ = 1/\tau$.*

Note, however, that this graded additive composition law is not only noncommutative, e.g., $(2\tau + 2) + \tau(3\tau + 2) \neq (3\tau + 2) + \tau^2(2\tau + 2)$. It is also non-associative, e.g.

$$(\tau + 1 + \tau^2(5\tau + 4)) + \tau(2\tau + 2) \neq (\tau + 1) + \tau^2(5\tau + 4 + \tau(2\tau + 2)). \quad (14)$$

Thus, indexing associative algebras over the quasicrystal chain is incompatible with this grading.

³This provides a recursive construction of the chain, starting from, e.g., $m = 1$, provided, however, that the parities of each new element were at hand. Empirically, one may check that the odd class is indexed by non-contiguous integers: $n_2 = 0, 2, 5, 7, 10, \dots$; while the even class is more populous: $n_2 = 1, 3, 4, 6, 8, 9, \dots$)

To recover associativity, one must consider a finer grading for the additive composition law of indices. Consider, instead, given an integer a , those points $n \in \Sigma((0, 1])$ s.t., $\forall m \in \Sigma((0, 1])$, one has $n + \tau^a m \in \Sigma((0, 1])$. In this case, we may say that n is *compatible with grading a* . Evidently, $n^* \in (0, 1 - \frac{1}{\tau^a}]$, for a even; and $n^* \in (\frac{1}{\tau^a}, 1]$, for a odd. Further note that, since the two intervals grow and overlap increasingly with growing a , a given index point n may be assigned *several even and odd* gradings a , for sufficiently large a . This multi-graded additive composition law also produces elements in $\Sigma((0, 1])$:

Lemma III.4 *For all $n \in \Sigma((0, 1])$, compatible with grading a and for all $m \in \Sigma((0, 1])$, compatible with grading b , the point $n + \tau^a m \in \Sigma((0, 1])$ by assumption) is compatible with grading $a + b$.*

Proof. There are four cases corresponding to the four possibilities for the gradings a, b to be even or odd; in self-evident notation,

$$\begin{aligned}
(1) \quad & n^* + \tau'^{2r+1} m^* \in [\frac{1}{\tau^{2r+1}}, 1] - [\frac{1}{\tau^{2r+1}} \frac{1}{\tau^{2s+1}}, 1] \in (0, 1 - \frac{1}{\tau^{2(r+s+1)}}], \\
(2) \quad & n^* + \tau'^{2r+1} m^* \in [\frac{1}{\tau^{2r+1}}, 1] - \frac{1}{\tau^{2r+1}} [0, 1 - \frac{1}{\tau^{2s}}] \in (\frac{1}{\tau^{2(r+s)+1}}, 1], \\
(3) \quad & n^* + \tau'^{2r} m^* \in [0, 1 - \frac{1}{\tau^{2r}}] + \frac{1}{\tau^{2r}} [\frac{1}{\tau^{2s+1}}, 1] \in (\frac{1}{\tau^{2(r+s)+1}}, 1], \\
(4) \quad & n^* + \tau'^{2r} m^* \in [0, 1 - \frac{1}{\tau^{2r}}] + \frac{1}{\tau^{2r}} [0, 1 - \frac{1}{\tau^{2s}}] \in (0, 1 - \frac{1}{\tau^{2(r+s)}}].
\end{aligned} \tag{15}$$

■

In contrast to the previous additive composition law, this one *is* manifestly associative, provided the grading of a “sum” is taken to be the sum of the gradings of the “summands”, as here.

Moreover, the set $\Sigma((0, 1])$ possesses *self-similarity* properties:

Lemma III.5 *Let $a \in \mathbb{N}$. Then*

$$\tau^a \Sigma((0, 1]) = \begin{cases} \Sigma([\frac{-1}{\tau^a}, 0)) & \text{for } a \text{ odd} \\ \Sigma((0, \frac{1}{\tau^a}]) & \text{for } a \text{ even} \end{cases} . \tag{16}$$

This implies that the sequence of consecutive points in $\Sigma((0, 1])$ corresponds to that in $\Sigma((0, \frac{1}{\tau^a}])$ (respectively to $\Sigma([\frac{-1}{\tau^a}, 0))$), up to rescaling of all nearest neighbour distances (distances between adjacent points in the set) by τ^a .

Proof. Let $n \in \tau^a \Sigma((0, 1])$. Then, $n = \tau^a m$ with $m^* \in (0, 1]$. Hence $n^* = (\tau')^a m^* \in (0, (\tau')^a]$ for a even; and $n^* \in [(\tau')^a, 0)$ for a odd. This is equivalent to $n \in \Sigma((0, (\tau')^a])$ for a even and $n \in \Sigma([\tau'^a, 0))$ for a odd. By the identity $\tau\tau' = -1$, the claim follows. ■

Hence, the decomposition (9) can be refined using

$$\Sigma((0, 1]) = (-\tau)\Sigma((0, 1]) \oplus \Sigma((\frac{1}{\tau}, 1]). \quad (17)$$

(17) again reflects the self-similarity properties of $\Sigma((0, 1])$: the set contains a copy of itself in which all distances are rescaled by τ and the points are reflected at the origin. The properties of $\Sigma((0, 1])$ listed in this section have implications on the consistent definition and structure of the algebras (5), as discussed in the next section. For example, the selfsimilarity properties lead to a class of subalgebras indexed by a set that is selfsimilar to the original index set.

4 Associativity of the algebras

In order to obtain a consistently associative product (5), the finer grading additive composition law of index points of Lemma III.4 is utilized. $\forall n \in \Sigma((0, 1])$ compatible with integer nonzero grading a , the algebra operators J_n^a are defined with the additive super-index a as the grading of the graded-additive point n in the sub-index.

The product $J_m^a J_n^b = J_{m+\tau^a n}^{a+b}$ is thus well defined, with the respective indices of the right-hand side in their proper set. Moreover, the resulting product is manifestly associative, as the graded composition of the lower indices is such:

$$J_n^a (J_m^b J_k^c) = J_n^a J_{m+\tau^b k}^{b+c} = J_{n+\tau^a m+\tau^{a+b} k}^{a+b+c}, \quad (18)$$

whilst the same holds for the alternative association order,

$$(J_n^a J_m^b) J_k^c = J_{n+\tau^a m}^{a+b} J_k^c = J_{n+\tau^a m+\tau^{a+b} k}^{a+b+c}. \quad (19)$$

Associativity follows directly from the compatibility of $n + \tau^a m$ with the additive grading $a + b$ of Lemma III.4.

Remark IV.1 *In practice, assigning allowable gradings to a given sub-index is a technical problem which has not been studied systematically—beyond the case-by-case specific evaluation of interval inclusion of the Galois conjugate of the sub-index, or the recursive fine-graded addition of starting points compatible with gradings 1 and 2 to build up further compatible gradings.*

With this it is possible to consistently introduce a Lie algebra over the set $\Sigma([-1, 1])$ as follows. For J_m^a with either $m \in \Sigma([-1, 0) \cup (0, 1])$ and $a \in \mathbb{N}$, or, $m = 0$ and $a = 0$,

$$[J_m^a, J_n^b] = J_{m+\tau^a n}^{a+b} - J_{n+\tau^b m}^{a+b} \quad (20)$$

forms a Lie algebra, denoted as $\mathcal{L}([-1, 1])$. Since the structure of the commutator is such that the lower central series does not become zero, the Lie algebra is not nilpotent, in contrast to the Lie algebra in (6).

The center of the Lie algebra is given by J_0^0 . No other element is central since the fine grading $a = 0$ is not permissible in combination with $m \neq 0$.

$\mathcal{L}([-1, 1])$ allows for a triangular decomposition. Indeed, with the notation $\mathcal{L}(\Omega)$ corresponding to the subalgebra with index set $\Sigma(\Omega)$ one has

$$\begin{aligned} \mathcal{L}([-1, 1]) &= \mathcal{L}([-1, 0)) \oplus \{J_0^0\} \oplus \mathcal{L}((0, 1]) \\ &= -\mathcal{L}((0, 1]) \oplus \{J_0^0\} \oplus \mathcal{L}((0, 1]), \end{aligned} \quad (21)$$

and the two copies $\mathcal{L}((0, 1])$ and $\mathcal{L}([-1, 0))$ are related by the automorphism that maps $m \in \Sigma((0, 1])$ on $-m \in \Sigma([-1, 0))$. This opens up the possibility to study highest weight representations for (4).

Independently of this, a simple formal operator realization of this algebra is [5]

$$J_m^a = e^{m \exp(x)} \tau^{a \partial_x}. \quad (22)$$

Subalgebras of (4) are of particular interest because they reflect the self-similarity properties of the index set. In particular, by virtue of (16), there exists a family of subalgebras, parameterized by $a \in \mathbb{N}$,

$$\mathcal{L}([-|\tau'|^a, |\tau'|^a]) \subset \mathcal{L}([-1, 1]) \quad (23)$$

with index sets selfsimilar to that of $\mathcal{L}([-1, 1])$, i.e. with index sets corresponding to copies of $\Sigma([-1, 1])$ rescaled by τ^a . This property distinguishes the Lie algebras in (4) from previously defined Lie algebras over aperiodic point sets, and makes them a generic object for the study of quasicrystals.

Conclusion

A closer inspection of the algebras (4) reveals that they are in fact representatives of a new type of infinite dimensional Lie algebra induced by semi-direct product laws. To see this, note that by virtue of the relation

$$\tau^n = f_n \tau + f_{n-1}, \quad n > 0, \quad (24)$$

where f_n denotes the n 'th Fibonacci number ($f_0 = 0$, $f_1 = 1$), one can express $m + \tau^a n = (m_1 + \tau m_2) + \tau^a (n_1 + \tau n_2)$ as $(m_1 + f_a n_2 + f_{a-1} n_1) + \tau(m_2 + f_a n_1 + f_a n_2 + f_{a-1} n_2)$. In particular, this induces a semi-direct product

$$\Sigma(\Omega) \times_{\Phi} \mathbb{N} := \{(\hat{m}, a) \mid \hat{m} = (m_1, m_2), m = m_1 + \tau m_2 \in \Sigma(\Omega), a \in \mathbb{N}\} \quad (25)$$

with

$$(\hat{m}, a) \times_{\Phi} (\hat{n}, b) = (\hat{m} + \Phi(a)\hat{n}, a + b) \quad (26)$$

where, using the identity $f_{a+1} = f_a + f_{a-1}$,

$$\Phi(a) = \begin{pmatrix} f_{a-1} & f_a \\ f_a & f_{a+1} \end{pmatrix}. \quad (27)$$

In this notation, the Lie algebra $\mathcal{L}([-1, 1])$ in (4) can equivalently be expressed as follows⁴:

$$[J_{(\hat{m}, a)}, J_{(\hat{n}, b)}] = J_{(\hat{m}, a) \times_{\Phi} (\hat{n}, b)} - J_{(\hat{n}, b) \times_{\Phi} (\hat{m}, a)}. \quad (28)$$

⁴Note that the Jacobi identity is fulfilled due to the identity $\Phi(a)\Phi(b) = \Phi(a+b)$ for any $a, b \in \mathbb{N}$.

It is called the *semi-direct product induced* Lie algebra $\mathcal{L}(\Sigma(\Omega) \times_{\Phi} \mathbb{N})$.

This opens up a new direction in the exploration of infinite dimensional Lie algebras. In particular, Lie algebras as in (4) can be associated also to other types of semidirect products, which do not necessarily need to be related to the aperiodic structures $\Sigma(\Omega)$. For example, the cyclotomic case [5] may also be viewed from this perspective.

The semi-direct product induced Lie algebras introduced here for (25) reflect properties of the aperiodic point sets such as their self-similarity. They hence lend themselves for applications in areas where aperiodicity plays a crucial role, such as for example in the study of mathematical and physical properties of quasicrystals [8]. They form a Lie algebra of Kac-Moody type, which closes on the points of a one-dimensional quasicrystal. This immediately raises the prospect of further extensions of this idea to other aperiodic sets, and also to semi-direct products related to the points of higher dimensional aperiodic sets, such as Penrose tilings [9]. From the point of view of physics it is this last possibility which holds the most promise of applications to the quantum mechanics of aperiodic lattices. In particular there is the prospect of constructing a field theory over the points of an aperiodic chain.

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